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VII—The Götz Inversion of Intensity-Ratio in Zenith-Scattered Sunlight

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1—Introduction

Götz, Meetham, and Dobson* have recently studied the distribution of ozone in the earth's atmosphere by a new method, comparing the intensities, at two wave-lengths, of ultra-violet scattered zenith-light. Regener† in a much more direct manner has found strikingly similar results, using a spectrograph carried high into the stratosphere by a balloon.

The zenith-light method has great importance for the study of the ozone-distribution, and may be applicable, in modified form, to other atmospheric problems, possibly including problems of radio waves scattered and absorbed in the ionosphere. But it is laborious and in some respects difficult to determine the ozone-distribution from the zenith-light measurements. It seems desirable, and the need has been urged upon me by Dr. Dobson, to examine further some of the mathematical questions involved in the method. This is attempted in the present paper, subject to the limitation that only primary scattered light is considered. The effect of secondary scattered light remains to be discussed in a further paper.

The interpretation of their work by Götz, Meetham, and Dobson has been questioned by Pekeris.‡ His criticism is here shown to be invalid (7.2, 7.21).

The plan of this paper is to consider certain special distributions of ozone for which the mathematical analysis can be carried to an advanced stage, though in some cases numerical integration must be resorted to at the final stage. In most of the cases considered, but not all, the air density is supposed to vary exponentially with the height. Complete numerical results have been calculated in a number of cases, mainly referring to an atmosphere on a flat earth ; but some cases relating to a spherical earth have been worked out numerically, and these suffice to show, in a general way, how the curvature of the earth affects the results.

* 'Proc. Roy. Soc.,' A, vol. 145, p. 416 (1934).

† E. Regener and V. H. Regener, 'Phys. Z.,' vol. 35, p. 788 (1934).

‡ 'Avh. norske VidenskAkad.,' 1933, No. 8.

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2—Notation

M, M_o = the total mass of air and ozone respectively, in the whole of a vertical column of unit cross-section extending to the limit of the atmosphere from the level of the instrument, at a point A.

z = the sun's zenith-distance from A, *i.e.*, its angular distance from the vertical at A.

$Z = \sec z - 1$.

H = the height of the homogeneous atmosphere at A, *i.e.*, the height of the atmosphere of *uniform* density equal to the density at A, and of the same total mass M.

This height H is taken as the *unit* of height and distance, in which the following lengths (amongst others) are reckoned.

r = the earth's radius from its centre O to A.

x or y = the height of any point P above A.

$\rho(y)$, $\rho_o(y)$ = the density of air or ozone respectively at P.

$m(y)$, $m_o(y)$ = the fraction of the air or ozone respectively which lies above P.

It follows from the definition of H, and its use as unit of length, that

$$M = \rho(0)\,H = \rho(0) \tag{1}$$

The following results follow from the definitions of M, M_o, ρ, ρ_o, m, m_o :—

$$M = \int_0^\infty \rho(y)\,dy, \quad M_o = \int_0^\infty \rho_o(y)\,dy, \tag{2}$$

$$m(y) = \frac{1}{M}\int_y^\infty \rho(x)\,dx, \quad m_o(y) = \frac{1}{M_o}\int_y^\infty \rho_o(x)\,dx, \tag{3}$$

$$m(0) = 1,\ m(\infty) = 0, \quad m_o(0) = 1,\ m_o(\infty) = 0, \tag{4}$$

$$\rho(y) = -M\frac{dm}{dy}, \quad \rho_o(y) = -M_o\frac{dm_o}{dy}, \tag{5}$$

whence, since m_o is clearly a function of m, we find for the ozone-concentration ρ_o/ρ the formula

$$\frac{\rho_o(y)}{\rho(y)} = \frac{M_o}{M}\frac{dm_o}{dm}. \tag{6}$$

2·1. *Uniformly Exponential Atmosphere*—In the special case of a "uniformly exponential" atmosphere, defined as one in which the density is everywhere the same exponential function of the height, so that

$$\rho(y) = \rho(0)\,e^{-\kappa y} \tag{1}$$

we have, by 2 (1), (3)

(2) $$m(y) = \frac{1}{\kappa} e^{-\kappa y},$$

and since $m(0) = 1$, it follows that $\kappa = 1$, and

(3) $$\rho(y) = \mathrm{M}\, e^{-y}, \quad m(y) = e^{-y}.$$

2.11. If the ozone distribution is also uniformly exponential, so that

(1) $$\rho_o(y) = \rho_o(0)\, e^{-ky},$$

we have

(2) $$\mathrm{M}_o = \rho_o(0)/k$$

(3) $$\rho_o(y) = k\, \mathrm{M}_o\, e^{-ky}, \quad m_o(y) = e^{-ky}.$$

2.12. If the atmosphere is distributed exponentially (with the same index) only over a certain range of height, y_1 to y_2, corresponding to a range m_1 to m_2 for m, in which

(1) $$\rho(y) = \mathrm{M}\kappa\mathrm{C}\, e^{-\kappa y},$$

it follows that

(2) $$m(y) = \mathrm{C}\, e^{-\kappa y} + \mathrm{D},$$

where C, D may readily be expressed in terms of m_1 and m_2.

2.13. If, for example,

$$\rho(y) = \mathrm{M}\kappa_1\mathrm{C}_1\, e^{-\kappa_1 y}, \qquad y \geq y_1$$

$$\rho(y) = \mathrm{M}\kappa\mathrm{C}\, e^{-\kappa y}, \qquad y_1 \geq y \geq y_2$$

$$\rho(y) = \mathrm{M}\kappa_2\mathrm{C}_2\, e^{-\kappa_2 y}, \qquad y_2 \geq y \geq 0,$$

for continuity of ρ at y_1 and y_2 it is necessary that

$$\kappa_1\mathrm{C}_1\, e^{-\kappa_1 y_1} = \kappa\mathrm{C}\, e^{-\kappa y_1},$$

$$\kappa\mathrm{C}\, e^{-\kappa y_2} = \kappa_2\mathrm{C}_2\, e^{-\kappa_2 y_2}.$$

In this case

$$m = \mathrm{C}_1\, e^{-\kappa_1 y}, \qquad y \geq y_1$$

$$m = \mathrm{C}\, e^{-\kappa y} + \mathrm{D}, \qquad y_1 \geq y \geq y_2$$

$$m = 1 - \mathrm{C}_2\,(1 - e^{-\kappa_2 y}), \qquad y_2 \geq y \geq 0$$

where for continuity of m at y_1 and y_2 we must have

$$\mathrm{D} = \mathrm{C}_1\, e^{-\kappa_1 y_1} - \mathrm{C}\, e^{-\kappa y_1},$$

$$\mathrm{C}\, e^{-\kappa y_2} + \mathrm{D} = 1 - \mathrm{C}_2\,(1 - e^{-\kappa_2 y_2}).$$

These relations suffice to determine C_1, C, C_2, and D for any given values of κ_1, κ, κ_2, y_1, and y_2.

2.2. The following definitions refer to solar radiation of a particular wave-length λ. Corresponding quantities relative to a second wave-length λ' will be denoted by the same symbols with an accent (′) attached. The suffix z will be used to indicate that the sun's zenith distance relative to the point A is z.

α, σ = absorption coefficient of ozone, and scattering coefficient of air, per unit mass.

a, s = total absorption coefficient of ozone, and total scattering coefficient of the air, for the vertical column above the level of A.

Thus

$a = \alpha M_o, \quad s = \sigma M,$

I_∞ = intensity of direct solar radiation at the outer limit of the atmosphere,

$I_{r,y,z}$ = intensity of direct solar radiation at the level y above A,

$I_{r,z}$ = total intensity at A of radiation scattered from the zenith,

$J_r = I_{r,z} \div \frac{1}{2}(1+\cos^2 z)\, I_{r,0},$

$R_r = J_r/J'_r = (I'_{r,0}/I_{r,0})\,(I_{r,z}/I'_{r,z}).$

3—General Formulæ

The direct radiation of intensity $I_{r,y,z}$ at height y is partly scattered downwards, and the scattered beam is attenuated by absorption during its vertical passage from P, at height y, to the instrument at A. Thus the intensity of scattered zenith light at A, derived from the height-interval $y, y+dy$, is

$$C_y \frac{3\sigma}{16\pi}(1+\cos^2 z)\, I_{r,y,z}\, \rho(y)\, dy, \tag{1}$$

where C_y is an attenuation factor defined by

$$C_y = \exp\left[-\int_0^y \{\alpha\rho_o(x) + \sigma\rho(x)\}\, dx\right]. \tag{2}$$

Thus, integrating over the whole range of y,

$$I_{r,z} = \frac{3\sigma}{16\pi}(1+\cos^2 z)\int_0^\infty C_y I_{r,y,z}\, \rho(y)\, dy. \tag{3}$$

3.1. To evaluate $I_{r,y,z}$ we must determine the total mass of air and ozone traversed in the passage of the light from outside the atmosphere to the point P. Consider any intermediate point P′ at height x on the path to P.

The sun's zenith distance λ at P′, relative to the vertical at P′, is given by

$$\sin\lambda = \frac{r+y}{r+x}\sin z.$$

The length of path dl of the ray between the heights $x + dx$, x is therefore sec $\lambda\, dx$. Consequently

$$I_{r,y,z} = I_\infty \exp\left[-\int_y^\infty \{\alpha\rho_o(x) + \sigma\rho(x)\} \sec\lambda\, dx\right].$$

3.11. *Formulæ and Integrals connected with* λ—The angle λ is clearly a function of $r + x$, $r + y$, and z, and when it is desired to indicate this it will be denoted by $\lambda_{r,x,y,z}$. The special case $r = \infty$ refers to a plane earth, for which $\lambda = z$ at all heights, *i.e.*,

$$\lambda_{\infty,x,y,z} = z.$$

For vertical incidence ($z = 0$) we have (for all values of r, x, y),

$$\lambda_{r,x,y,0} = 0.$$

We may note that

(1) $$\sec\lambda = \frac{r+x}{\sqrt{\{(r+x)^2 - (r+y)^2 \sin^2 z\}}}$$

and also

(2) $$\sec\lambda\, dx = -(r+y)\sin z \operatorname{cosec}^2\lambda\, d\lambda.$$

When $x = \infty$, $\lambda = 0$; when $x = y$, $\lambda = z$.

We shall later have occasion to consider integrals of the form

(3) $$\int_{y_1}^{y_2} e^{-kx} \sec\lambda\, dx.$$

These we shall consider as the difference between two integrals with upper limit ∞, *i.e.*,

$$\left(\int_{y_1}^\infty - \int_{y_2}^\infty\right) e^{-kx} \sec\lambda\, dx.$$

Thus we consider

$$\int_{y_1}^\infty e^{-kx} \sec\lambda\, dx.$$

Let us write

(4) $$\sin\zeta_1 = \frac{r+y}{r+y_1}\sin z.$$

Then as x goes from y_1 to ∞, λ goes from ζ_1 to 0. Hence

(5) $$\int_{y_1}^\infty e^{-kx}\sec\lambda\, dx = (r+y)\sin z \int_0^{\zeta_1} e^{-kx} \operatorname{cosec}^2\lambda\, d\lambda.$$

In this we write

$$e^{-kx} = \exp\left[-k\left\{(r+y)\frac{\sin z}{\sin\lambda} - r\right\}\right]$$
$$= \exp\left[-k\left\{(r+y_1)\frac{\sin\zeta_1}{\sin\lambda} - r\right\}\right]$$
$$= e^{-ky_1}\exp\left\{k(r+y_1)\left(1 - \frac{\sin\zeta_1}{\sin\lambda}\right)\right\}.$$

Hence

(6) $$\int_{y_1}^{\infty} e^{-kx} \sec \lambda \, dx$$

$$= e^{-ky_1} (r + y_1) \sin \zeta_1 \int_0^{\zeta_1} \exp \left\{ k (r + y_1) \left(1 - \frac{\sin \zeta_1}{\sin \lambda} \right) \right\} \operatorname{cosec}^2 \lambda \, d\lambda$$

$$\equiv \frac{1}{k} e^{-ky_1} f \{ k (r + y_1), \zeta_1 \},$$

where

(7) $$f(\mathrm{X}, \zeta) \equiv \mathrm{X} \sin \zeta \int_0^{\zeta} \exp \left\{ \mathrm{X} \left(1 - \frac{\sin \zeta}{\sin \lambda} \right) \right\} \operatorname{cosec}^2 \lambda \, d\lambda.$$

The function f is the absorption function which I introduced, discussed, and tabulated (over certain ranges of X and ζ) in a recent paper.* When ζ is small f is nearly equal to $\sec \zeta$, though it is always less than $\sec \zeta$ when $\zeta > 0$. Unlike $\sec \zeta$ it is finite at $\zeta = 90°$ and beyond, though it increases rapidly with ζ in this region, and tends to infinity as ζ increases. An extended table of f (X, ζ) is here given.

TABLE I

f (X, ζ)

$\zeta =$	30°	45°	60°	75°	80°	83°	85°	87°	90°	93°	95°
X=400	1·1537	1·4107	1·9854	3·7420	5·3797	7·2467	9·3295	12·8190	25·0898	73·9378	220·15
800	1·1542	1·4125	1·9926	3·7999	5·5514	7·6512	10·1445	14·7298	35·4657	197·4428	1476·16
1600	1·1545	1·4133	1·9963	3·8310	5·6496		10·7057		50·1443		44107·57
∞	1·1547	1·4142	2·0000	3·8637	5·7588	8·2055	11·4737	19·1073	∞	—	—

$f(800, 88°) = 18·6864$ $f(800, 92°) = 96·7531.$

Thus, with a definition of ζ_2 corresponding to that of ζ_1,

(8) $$\int_{y_1}^{y_2} e^{-kx} \sec \lambda \, dx = \frac{1}{k} [e^{-ky_1} f \{ k (r + y_1), \zeta_1 \} - e^{-ky_2} f \{ k (r + y_2), \zeta_2 \}].$$

In this formula ζ_1 is a function not only of y_1 and z but also of y; likewise for ζ_2. In particular

(9) $$\int_{y}^{\infty} e^{-kx} \sec \lambda \, dx = \frac{1}{k} e^{-ky} f \{ k (r + y), z \}.$$

3.2. Substituting from 3.1 for $\mathrm{I}_{r,y,z}$ in the formula 3 (3) for $\mathrm{I}_{r,z}$ we have

$$\mathrm{I}_{r,z} = \frac{3\sigma}{16\pi} (1 + \cos^2 z) \, \mathrm{I}_{\infty} \int_0^{\infty} \exp \left[- \int_y^{\infty} \{ \alpha \, \rho_o (x) + \sigma \, \rho (x) \} \sec \lambda \, dx \right.$$
$$\left. - \int_o^{y} \{ \alpha \, \rho_o (x) + \sigma \, \rho (x) \} \, dx \right] \rho (y) \, dy.$$

* 'Proc. Phys. Soc.,' vol. 43, p. 483 (1931).

For vertical incidence this reduces to

$$I_{r,0} = \frac{3\sigma}{8\pi} I_\infty e^{-a-s} M = \frac{3s}{8\pi} e^{-a-s} I_\infty.$$

Consequently

$$J_r \equiv I_{r,z} \div \tfrac{1}{2}(1+\cos^2 z)\, I_{r,0}$$

$$= \frac{1}{M}\int_0^\infty \exp\left[-\int_y^\infty \{\alpha\,\rho_o(x) + \sigma\,\rho(x)\}(\sec\lambda - 1)\,dx\right]\rho(y)\,dy.$$

In this expression we change the variable y to m, using 2 (5), and obtain

$$J_r = \int_0^1 \exp\left[-\int_y^\infty \{\alpha\rho_o(x) + \sigma\rho(x)\}(\sec\lambda - 1)\,dx\right] dm.$$

In the case of a plane earth ($r = \infty$), for which $\lambda = z$, writing

$$Z \equiv \sec z - 1,$$

we obtain, on integration of the inner integral in J_r,

$$J_\infty = \int_0^1 e^{-Z(am_o + sm)}\,dm.$$

4—J_∞ for Special Types of Atmosphere on a Plane Earth

4.1. *Constant Concentration of Ozone*—Suppose that at all heights

$$\rho_o = c\rho,$$

so that

$$M_o = cM,$$

and at all heights

$$m_o = m.$$

Then

$$J_\infty = \int_0^1 e^{-Z(a+s)m}\,dm = \frac{1 - e^{-Z(a+s)}}{Z(a+s)}.$$

4.2. Suppose that

$$m_o = m^k,$$

where $k > 0$. Then by 2 (6)

$$\frac{\rho_o}{\rho} = k\frac{M_o}{M} m^{k-1},$$

so that the ozone concentration increases upwards if $k < 1$, and downwards if $k > 1$. The preceding case, of constant ozone concentration, corresponds to $k = 1$.

For two special cases $k = \frac{1}{2}$ and $k = 2$, J_∞ is expressible in terms of known functions. Thus, write

$$F(a, s, Z, m_1) \equiv \int_{m_1}^{1} e^{-Z(am^2+sm)}\, dm$$
$$= \frac{1}{(aZ)^{\frac{1}{2}}} e^{s^2Z/4a} \left[\operatorname{erf}\left\{\left(1 + \frac{s}{2a}\right)(aZ)^{\frac{1}{2}}\right\} - \operatorname{erf}\left\{\left(m_1 + \frac{s}{2a}\right)(aZ)^{\frac{1}{2}}\right\}\right],$$

where

$$\operatorname{erf} x \equiv \int_0^x e^{-u^2}\, du.$$

The function erf x is tabulated over a considerable range of x, and when x is large can be calculated from the asymptotic formula

$$\operatorname{erf} x = \frac{\sqrt{\pi}}{2} - \frac{e^{-x^2}}{2x}\{1 + S(x)\},$$

where

$$S(x) = \sum_1^\infty (-1)^n \frac{1.3\ldots(2n-1)}{2^n x^{2n}}.$$

4.21. Thus when $k = 2$,

$$J_\infty = F(a, s, Z, 0).$$

4.22. When $k = \frac{1}{2}$,

(1) $$J_\infty = \int_0^1 \exp\{-Z(am^{\frac{1}{2}} + sm)\}\, dm.$$

Putting $m = u^2$, we have

(2) $$J_\infty = \frac{1}{s}\int_0^1 \exp\{-Z(au + su^2)\}(2su + a)\, du - \frac{a}{s}\int_0^1 \exp\{-Z(au + su^2)\}\, du$$
$$= \frac{1}{sZ}\{1 - e^{-Z(a+s)}\} - \frac{a}{s} F(s, a, Z, 0).$$

4.3. A more general integrable case is the following :—

(1) $$m_o = cm^k \quad \text{for} \quad 0 \leq m \leq m_1,\ y \geq y_1,$$

(2) $$m_o = 1 - d(1 - m) - g(1 - \sqrt{m}) \text{ for } m_1 \leq m \leq 1, y_1 \geq y \geq 0.$$

The latter is unity, as it must be, when $m = 1$. For continuity at y we have

$$cm_1^k = 1 - d(1 - m_1) - g(1 - \sqrt{m_1}).$$

The ozone concentration is given by

$$\frac{\rho_o}{\rho} = \frac{M_o}{M} kc\, m^{k-1} \qquad y \geq y_1,$$
$$\frac{\rho_o}{\rho} = \frac{M_o}{M}\left(d + \frac{g}{2\sqrt{m}}\right). \qquad y_1 \geq y \geq 0.$$

For continuity at y_1 we must have

$$kcm_1^{k-1} = d + g/2\sqrt{m_1}.$$

Hence, eliminating c, we get

$$d\left(1 - \frac{k-1}{k}\, m_1\right) = 1 - g\left\{1 - \left(1 - \frac{1}{2k}\right)\sqrt{m_1}\right\}.$$

Thus c and d can be expressed in terms of g, which, together with m_1, is at our disposal, and enables a considerable variety of ozone distributions to be represented.

In evaluating J, the integration over the ranges 0 to m_1 and m_1 to 1 must be separately considered. They are expressible in terms of known functions when $k = 1$ or $k = 2$.

4.31. When $k = 1$, we find that

$$J_\infty = \frac{1 - e^{-Z(ac+s)\,m_1}}{Z\,(ac+s)}$$

$$+ \frac{1}{Z\,(ad+s)}\left[e^{-Z\,[a - ag\,(1-\sqrt{m_1}) - ad\,(1-m_1) + sm_1]} - e^{-Z\,(a+s)}\right]$$

$$- \frac{ag}{ad+s}\, e^{-aZ\,(1-d-g)}\, F\,(ad+s,\, ag,\, Z,\, m_1).$$

When $m_1 = 0$, $d = 0$, and $g = 1$, this case reduces to that of 4.22.

4.32. When $k = 2$, we find that J_∞ is obtainable from the expression given in 4.31 by substituting for the first term the expression

$$F\,(ac,\, s,\, Z,\, 0) - F\,(ac,\, s,\, Z,\, m_1).$$

4.33. If the ozone is absent below the level y_1 at which $m = m_1$, then $m_o = 1$ for $m > m_1$. Suppose that above this level

$$m_o = (m/m_1)^k, \qquad\qquad 0 \leq m \leq m_1.$$

This is a special case of 4.3, corresponding to

$$c = 1/m_1^k, \qquad d = g = 0.$$

The special values assumed by J_∞ in the cases $k = 1$, $k = 2$ are as follows:—

$$k = 1 \qquad J_\infty = \frac{1}{Z}\left\{\frac{m_1}{a+sm_1} + \frac{a}{s(a+sm_1)}\, e^{-Z\,(a+sm_1)} - \frac{1}{s}\, e^{-Z\,(a+s)}\right\}$$

$$k = 2 \qquad J_\infty = F\,(a/m_1^2,\, s,\, Z,\, 0) - F\,(a/m_1^2,\, s,\, Z,\, m_1)$$

$$+ \frac{1}{Zs}\left\{e^{-Z\,(a+sm_1)} - e^{-Z\,(a+s)}\right\}.$$

4.34. Suppose that above the level $m = m_1$ the ozone is distributed so that

$$m_o = cm^k$$

while the remaining ozone, of amount $(1 - c\, m_1^k)\, M_o$, is concentrated at the level y_2 at which $m = m_2$. Then the first part of J_∞ is as given in 4.31, 4.32 for $k = 1$, $k = 2$. The second part, corresponding to the range $m = m_1$ to $m = m_2$, is

$$\frac{1}{Zs} \exp\left(- ac\, Z\, m_1^k\right) \{e^{-Zsm_1} - e^{-Zsm_2}\},$$

and the third part, corresponding to the range from m_2 to 1, is

$$\frac{1}{Zs} \{e^{-Z(a+sm_2)} - e^{-Z(a+s)}\}.$$

In this case the mean height of the ozone, if $k = 2$, is

$$\bar{y} = y_2 + (y_1 - y_2 + \tfrac{1}{2})\, cm_1^2.$$

4.41. A further case to be considered refers to an atmosphere such that, between two heights y_1 and y_2 $(< y_1)$

$$m = C\, e^{-\kappa y} + D \qquad y_1 \geq y \geq y_2$$

as in 2.12; thus between these limits the atmosphere is exponential,

$$\rho\,(y) = MC\kappa\, e^{-\kappa y}.$$

Outside these limits the air distribution is unrestricted.

The values of m at y_1 and y_2 will be denoted by m_1 and m_2, so that

$$m_1 = C\, e^{-\kappa y_1} + D, \quad m_2 = C\, e^{-\kappa y_2} + D.$$

If the atmosphere is uniformly exponential, $C = 1$, $\kappa = 1$, $D = 0$.

The ozone distribution to be considered is as follows :—

$$m_o = cm^k \qquad y \geq y_1, \quad 0 \leq m \leq m_1$$

$$m_o = cm_1^k + (1 - cm_1^k)\frac{y_1 - y}{y_1 - y_2} \qquad y_1 \geq y \geq y_2, \quad m_1 \leq m \leq m_2$$

$$m_o = 1. \qquad y_2 \geq y \geq 0, \quad m_2 \leq m \leq 1$$

Thus, by 2 (5), ρ_o is constant between y_1 and y_2, and zero below y_2; the constant value is

$$(1 - cm_1^k)\,\frac{M_o}{y_1 - y_2}.$$

We consider two values of k as before; the first part of J_∞, corresponding to the range of integration 0 to m_1, is the same as the first term in 4.31 if $k = 1$; if $k = 2$, as in 4.32, it is $F(ac, s, Z, 0) - F(ac, s, Z, m_1)$.

The part of J_∞ corresponding to the integration from m_2 to 1 is

$$\frac{1}{Zs}\{e^{-Z(a+sm_2)} - e^{-Z(a+s)}\}.$$

The part of J_∞ corresponding to the integration from m_1 to m_2 is

$$\frac{(CZs)^\beta}{Zs}\exp\left[-\frac{Za}{y_1-y_2}(y_1 - cm_1{}^k y_2) - ZsD\right]\int_{Zs(m_1-D)}^{Zs(m_2-D)} e^{-u}u^{-\beta}\,du,$$

where

$$\beta = \frac{Za(1-cm_1{}^k)}{\kappa(y_1-y_2)}.$$

The integral here is an incomplete gamma function, which is tabulated for certain values of β and of the limits.

4.42. When the constant ozone density extends down to the level $y = 0$, *i.e.*, when $y_2 = 0$, we have $m_2 = 1$, $C + D = 1$; the third part of J_∞ now vanishes, and the second part becomes

$$C^\beta (Zs)^{\beta-1} e^{-Z(a+D)} \int_{Zs(m_1-D)}^{Zs(1-D)} e^{-u} u^{-\beta}\,du.$$

4.43. If the ozone is absent above y_1, we have $c = 0$, and

$$(1)\quad J_\infty = \frac{1}{Zs}\left[1 - e^{-Zsm_1} + e^{-Z(a+sm_2)} - e^{-Z(a+s)} + (CZs)^\beta \exp\left\{-\frac{Zay_1}{y_1-y_2} - ZsD\right\}\int_{Zs(m_1-D)}^{Zs(m_2-D)} e^{-u}u^{-\beta}\,du\right]$$

where now

$$(2)\qquad \beta = \frac{Za}{\kappa(y_1-y_2)}.$$

If also $y_2 = 0$, so that the ozone has constant density from $y = 0$ to $y = y_1$, we have $m_2 = 1$, $C + D = 1$, $\beta = Za/\kappa y_1$,

$$(3)\qquad J_\infty = \frac{1}{Zs}\left[1 - e^{-Zsm_1} + (CZs)^\beta e^{-Z(a+sD)}\int_{Zs(m_1-D)}^{Zs(1-D)} e^{-u}u^{-\beta}\,du\right].$$

If the atmosphere is uniformly exponential, $C = 1$, $\kappa = 1$, $D = 0$, $\beta = Za/y_1$, and

$$(4)\qquad J_\infty = \frac{1}{Zs}\left[1 - e^{-Zsm_1} + (Zs)^\beta e^{-Za}\int_{Zsm_1}^{Zs} e^{-u}u^{-\beta}\,du\right].$$

5—R_∞ for $z = 90°$ for Special Types of Atmosphere on a Plane Earth

We now consider the limiting value of R_∞ as $z \to 90°$, for the special types of atmosphere on a plane earth considered in § 4. As $z \to 90°$, $Z \to \infty$. In this case, if $m_1 < 1$,

$$(1) \qquad F(a, s, Z, m_1) \sim \frac{e^{-m_1 sZ - m_1{}^2 aZ}}{Z(2m_1 a + s)} \left\{1 - \frac{2a}{(2m_1 a + s)^2 Z} + \ldots\right\}.$$

In particular,

$$(2) \qquad F(a, s, Z, 0) \sim \frac{1}{Zs}\left\{1 - \frac{2a}{s^2 Z} + \ldots\right\}.$$

5.1. *Constant Concentration of Ozone*—It is readily seen that in this case

$$R_\infty \to \frac{a' + s'}{a + s}.$$

5.21. When $m_o = m^2$, it follows from 4.21 and 5 (2) that

$$R_\infty \to \frac{s'}{s}.$$

5.22. When $m_o = m^{\frac{1}{2}}$, it follows from 4.22 (2) and 5 (2) that

$$R_\infty \to \frac{a'^2}{a^2} = \frac{\alpha'^2}{\alpha^2}.$$

5.31. When $m_o = cm$ above y_1, while below y_1 it has the form 4.3 (2), then whatever the values of d and g, we find

$$R_\infty \to \frac{a'c + s'}{ac + s}.$$

This applies also to the case of 4.33 ($k = 1$, $c = 1/m_1$), in which there is no ozone below y_1, and to that of 4.4 when $k = 1$.

5.32. When $m_o = cm^2$ above y_1, while below y_1 it has the form 4.3 (2), then whatever the values of d and g we find

$$R_\infty \to \frac{s'}{s} = \frac{\sigma'}{\sigma}.$$

This applies also to the case of 4.33 ($k = 2$), in which there is no ozone below y_1, and to that of 4.4 ($k = 2$).

5.4. It is, in fact, not difficult to show that, if $m_o = cm^k$ above y_1, then whatever the distribution of ozone below this limit, $R_\infty \to \dfrac{a'c + s'}{ac + s}$ (of which $\dfrac{a' + s'}{a + s}$ is the special case for $c = 1$), if $k = 1$, while if $k > 1$, $R_\infty \to s'/s$, depending only on the total scattering coefficients of the air. The latter case, of course, corresponds to

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an ozone concentration which decreases to zero at infinite height, proportionally to m^{k-1}, or, if the atmosphere is exponential (at least at great heights) to $e^{-(k-1)y}$.

The case $k < 1$ corresponds to an ozone concentration increasing indefinitely upwards. I have shown* that such a distribution is improbable, and that, in fact, above a certain height the concentration is likely to decrease. Thus we may expect that (on a plane earth) the limiting value R_∞ will be s'/s.

6—J_r for Special Ozone Distributions in an Exponential Atmosphere on a Curved Earth

When the curvature of the earth is taken into account, it is difficult to obtain J_r in terms of known functions, and especially so unless the air density is distributed exponentially. Hence only exponential atmospheres will be considered in the present case.

If the atmosphere is uniformly exponential, then

$$\rho(y) = Me^{-y},\ m = e^{-y},$$

so that

$$\int_y^\infty \sigma\,\rho(x)(\sec\lambda - 1)\,dx = s\int_y^\infty e^{-x}\sec\lambda\,dx - sm$$

$$= sm\,[f(r+y, z) - 1] = sm\,[f(r - \log m, z) - 1]$$

by 3.11.

6.1 *Uniformly Exponential Ozone Distribution*—Suppose that at all heights

(1) $$\rho_o(y) = kM_o\,e^{-ky},$$

so that

(2) $$m_o = e^{-ky} = m^k.$$

Then

(3) $$\int_y^\infty \alpha\,\rho_o(x)(\sec\lambda - 1)\,dx = ka\int_y^\infty e^{-kx}(\sec\lambda - 1)\,dx$$
$$= am^k\,[f\{k(r - \log m), z\} - 1],$$

by 3.11. Thus

(4) $$J_r = \int_0^1 \exp\left\{-am^k\,[f\{k(r-\log m), z\} - 1] - sm\,[f(r - \log m, z) - 1]\right\} dm.$$

6.11. The values of r that are of interest in connection with the actual atmosphere (in the region where the ozone occurs) may be taken to lie between 400 and 1000; actually r is 6370 km, while the unit in which it is here measured is $H_1 = kT/mg$, where k is Boltzmann's constant $1{\cdot}372 \,.\, 10^{-16}$, m is the mean molecular weight of the air, g is the acceleration of gravity, and T is the absolute temperature. Of course k, m, g have no connection with the k, m, g occurring elsewhere in the formulæ

* 'Phil. Mag.,' vol. 10, p. 369 (1930).

of this paper. Taking $k/m = 2{\cdot}87 \,.\, 10^6$*, $H = 2{\cdot}92 \,.\, 10^3$ T cm. This is 8·76 km for T = 300°, or 6·43 km for T = 220°; the corresponding values of r are 727 and 991; $r = 400$ corresponds to T = 584°.

6.12. For values of z near 90°, the function f is large, and the integrand of 6.1 (4) is very small except near the origin. In this region $-\log m$ becomes large, but $r - \log m$ differs significantly from r, so far as concerns the value of the functions f in 6·1 (4), only for extremely low values of m. It is, therefore, possible and convenient to approximate to J_r by omitting the term $-\log m$ in the functions f, which then become constant so far as the integration is concerned. When $k = 1, \frac{1}{2}$ or 2 the integration can be performed as in 4.2—4.22.

6.21. Thus when $k = 1$ we have

$$J_r = \frac{1 - \exp\{-(a+s)(f-1)\}}{(a+s)(f-1)}$$

$$R_r = \frac{a'+s'}{a+s}\,\frac{1-\exp\{-(a+s)(f-1)\}}{1-\exp\{-(a'+s')(f-1)\}}.$$

The second formula will be an even better approximation than the first, because J and J′ are both affected in a similar way by the approximation to f, which in these two formulæ denotes $f(r, z)$.

If we write

$$f(r, z) = \sec z_1,$$

then in the present case

$$R_r(z) = R_\infty(z_1),$$

so that the value of R_r for the curved earth for the zenith distance z corresponds to the value of R_∞ for a plane earth at the smaller zenith distance z_1.

As in 5.1, it is clear that as z increases (beyond 90°) and $f \to \infty$, R_r tends to the limiting value $(a'+s')/(a+s)$.

6.22. When $k = 2$, let f denote $f(r, z)$, as before, and let

$$\theta_{r,z} = \frac{f(r,z) - 1}{f(2r,z) - 1}.$$

Clearly $\theta_{r,z} < 1$. Then, as in 4.21,

$$J_r = F(a/\theta_{r,z}, s, f-1, 0),$$

which is equal to the value of J_∞ for the angle z_1 instead of z, for an ozone distribution increased at all heights in the ratio $1/\theta_{r,z}$.

As in 5.21, it is possible to show that in the present case $R_r \to s'/s$ as z increases beyond 90° and $f \to \infty$.

* JEANS, "Dynamical Theory of Gases," p. 119.

6.23. When $k = \frac{1}{2}$ we find, as in 4.22,

$$J_r = \frac{1}{s(f-1)}\left[1 - \exp\{-(a+s)(f-1)\}\right] - \frac{a}{s\theta_{\frac{1}{2}r,\,z}} F(s, a/\theta_{\frac{1}{2}r,\,z}, f-1, 0).$$

6.24. The error involved in replacing $f(r - \log m, z)$ by $f(r, z)$ in the above formulæ will now be considered. This can be done most simply for the case when $z = \frac{1}{2}\pi$, and since the error appears to increase with z, it will be sufficient if it can be proved negligible for this rather extreme value of z.

As shown in my paper (p. 487) already cited on p. 210, $f(r, 90°)$ is approximately $(\frac{1}{2}\pi r)^{\frac{1}{2}}$, the error decreasing as r increases, and being less than 1% when $r = 50$.

It may at the outset be noted that, as $m \to 0$, $f(r - \log m, z) \to \sec z$, so that certainly $mf \to 0$ for any value of z other than $\frac{1}{2}\pi$; also for this value it is clear from the above approximation that $mf(r - \log m, 90°) \to 0$ as $m \to 0$. Similar remarks apply to $f\{k(r - \log m), z\}$. Thus the integrand of J_r is finite over the whole range $m = 0$ to $m = 1$.

Since in J_r we are concerned with the integral of an exponential, in making the above approximation we ignore a *factor* in the integrand, this factor being, as we shall show, very nearly unity. Thus, consider the factor due to the approximation to the second part of the exponential in 6.1 (4), for $z = \frac{1}{2}\pi$; it is $\exp\{sm(\pi/8r)^{\frac{1}{2}} \log m\}$. The maximum value of $-m \log m$ is $1/e$, for the value $m = 1/e$. Thus the least value of the factor is $\exp\{-(s/e)\sqrt{(\pi/8r)}\}$, and if s is 1 or less, and r is 800, the factor is nearer to unity than 0·9915. The factor due to the earlier term, taking $k = 2$, is $\exp\{am^2 \log m \sqrt{(\pi/16r)}\}$, and the maximum value of $-m^2 \log m$ is $1/2e$, for $m = 1/\sqrt{e}$. Thus for other values of m the corresponding factor is nearer to unity than $\exp\{-(a/2e)\sqrt{(\pi/16r)}\}$, and if $a < \frac{1}{2}$, $r = 800$, this is not less than 0·9986. The product of the two factors is at least 0·9905, so that the integrand of J is nowhere in error by more than 1% owing to the approximations to f; this estimate of 1% is in excess, because the two terms in J do not attain their maxima at the same value of m; further, when z is near $\frac{1}{2}\pi$ the main part of J arises from a range of m much nearer the origin than the values $m = 1/e$ or $m = 1/\sqrt{e}$, and over this range the error factor is much nearer to unity than 0·99. Thus J is obtained as above to an accuracy well within 1%.

The value of R_r is still less affected by the approximation to f. This is strikingly illustrated by the results given later in 7.1, 7.11, 7.2, where it is seen that the variation of r from 400 to 800 to ∞ (changing mf and m^2f far more than is done by the above approximations) only affects R_r by a fraction of 1%.

6.3. Suppose the ozone is entirely above the level y_1, where it is distributed so that

$$m_o = cm^k = ce^{-ky}, \quad \rho_o = kcM_o e^{-ky},$$

where $c = m_1^{-k}$ since m_o must in this case be unity when $m = m_1$. Then if $y > y_1$,

$$\int_y^\infty \alpha\rho_o(x)(\sec\lambda - 1)\,dx = ack\int_y^\infty e^{-kx}(\sec\lambda - 1)\,dx$$
$$= ace^{-ky}[f\{k(r+y), z\} - 1],$$

by 3.11 (9). If $y < y_1$, since $\rho_o = 0$ below y_1, we have

$$\begin{aligned}\int_y^\infty \alpha\rho_o(x)(\sec\lambda - 1)\,dx &= \int_{y_1}^\infty \alpha\rho_o(x)(\sec\lambda - 1)\,dx\\ &= ack\int_{y_1}^\infty e^{-kx}(\sec\lambda - 1)\,dx\\ &= ace^{-ky_1}[f\{k(r+y_1),\zeta_1\} - 1],\end{aligned}$$

by 3.11 (6), where ζ_1 is a function of r, y_1, z, and y given by 3.11 (4). It is always less than 90°, even though z may exceed 90°; also it exceeds z_1, given by $\sin z_1 = r\sin z/(r+y_1)$.

When $k = 1, \frac{1}{2}$ or 2, the integration in J_r can be performed as before (after substituting for c, and making the usual approximation to f), but only over the range $m = 0$ to $m = m_1$; the fact that ζ_1 is a function of y, and therefore of m, complicates the corresponding integration over the rest of the range. The first part of J_r is

$$\frac{1}{(ac+s)(f-1)}[1 - \exp\{-m_1(ac+s)(f-1)\}]$$

when $k = 1$, in which case $c = 1/m_1$; when $k = 2$, and $c = 1/m_1^2$, it is

$$F(ac/\theta_{r,z}, s, f-1, 0) - F(ac/\theta_{r,z}, s, f-1, m_1).$$

The second part of J_r can be written

$$\frac{1}{s(f-1)}\exp[-a\{f(kr,\zeta) - 1\}][e^{-(f-1)sm_1} - e^{-(f-1)s}],$$

where ζ is intermediate between z_1 and z. The uncertainty as to its exact value is small when z is less than, and not too near, 90°; when z approaches 90° the second term becomes small and the uncertainty as to ζ, though now considerable, probably affects J_r very slightly.

6.31. Suppose that, as in 4.34, the ozone is distributed exponentially above y_1, where $m_o = cm^k$ (but $c \neq m_1^{-k}$ as in 6.3), while the remaining mass of ozone, $(1 - cm_1^k)M_o$, is concentrated at the level y_2 ($< y_1$), at which $m = m_2$. Then

$$\begin{aligned}&\int_y^\infty \alpha\rho_o(x)(\sec\lambda - 1)\,dx\\ &\quad= acm^k[f\{k(r - \log m), z\} - 1] && y \geq y_1\\ &\quad= acm_1^k[f\{k(r - \log m_1), \zeta_1\} - 1] && y_1 \geq y > y_2\\ &\quad= acm_1^k[f\{k(r - \log m_1), \zeta_1\} - 1]\\ &\qquad\quad + a(1 - cm_1^k)(\sec\zeta_2 - 1), && y_2 > y\end{aligned}$$

where

$$\sin\zeta_2 = \frac{r+y}{r+y_2}\sin z.$$

Thus ζ_2, like ζ_1, varies with y.

In this case J_r is divisible into three parts corresponding to the ranges of m from 0 to m_1, m_1 to m_2, and m_2 to 1. The first part is the same as in 6.3 (though the value of c is now different). The remainder of J_r is given, making the usual approximation to f, by

$$\frac{1}{s(f-1)}\Big\{\exp\left[-acm_1^k\{f(kr,\zeta)-1\}\right]\left[e^{-(f-1)sm_1}-e^{-(f-1)sm_2}\right]$$
$$+\exp\left[-acm_1^k\{f(kr,\zeta')-1\}-a(1-cm_1^k)(\sec\zeta''-1)\right]\left[e^{-(f-1)sm_2}-e^{-(f-1)s}\right]\Big\},$$

where ζ is intermediate between z and z_2, given by $\sin z_2=(r+y_2)\sin z/(r+y_1)$, while ζ' and ζ'' are intermediate between z and z' given by $\sin z'=r\sin z/(r+y_2)$; also f denotes, as elsewhere, $f(r,z)$.

6.4. We next consider the ozone distribution of 4.41. The atmosphere here considered being uniformly exponential, the C and κ of 4.41 are unity, while $D=0$.

The integral J is now divided, as in 4.41, into three parts, of which the first is the same as in 6.3.

In evaluating the second and third parts, we use the result, valid for $y_2<y<y_1$,

$$\left(\int_y^{y_1}+\int_{y_1}^{\infty}\right)\{\alpha\rho_o(x)+\sigma\rho(x)\}(\sec\lambda-1)\,dx=\frac{a(1-cm_1^k)}{y_1-y_2}\int_y^{y_1}(\sec\lambda-1)\,dx$$
$$+acm_1^k\left[f\{k(r-\log m_1),\zeta_1\}-1\right]+sm\left[f(r-\log m,z)-1\right],$$

while if $y<y_2$,

$$\left(\int_y^{y_2}+\int_{y_2}^{y_1}+\int_{y_1}^{\infty}\right)\{\alpha\rho_o(x)+\sigma\rho(x)\}(\sec\lambda-1)\,dx=\frac{a(1-cm_1^k)}{y_1-y_2}\int_{y_2}^{y_1}(\sec\lambda-1)\,dx$$
$$+acm_1^k\left[f\{k(r-\log m_1),\zeta_1\}-1\right]+sm\left[f(r-\log m,z)-1\right].$$

In evaluating the integral of $\sec\lambda-1$, we note that (*cf.* 3.11 (1))

$$\sec\lambda-1=\frac{dw}{dx},$$

where w (or, indicating the variables on which it depends $w_{r,x,y,z}$) is given by

$$w=\surd\{(r+x)^2-(r+y)^2\sin^2 z\}-x+y-(r+y)\cos z,$$

while

$$w_{r,x,x,z}=0,\qquad w_{r,x,y,0}=0.$$

Thus

$$\int_y^{y_1}(\sec\lambda-1)\,dx=w_{r,y_1,y,z}-w_{r,y,y,z}=w_{r,y_1,y,z},$$

$$\int_{y_2}^{y_1}(\sec\lambda-1)\,dx=w_{r,y_1,y,z}-w_{r,y_2,y,z}.$$

Consequently the index of the exponential in the integrand of J_r is now expressed completely in terms of known functions. But the final integration, over the range m_1 to 1, involves quadratures which in the present case seem inescapable.

6.41. If we take $c = 0$ in the formulæ of 4.41, 6.4, we get the case of an atmosphere in which there is a uniform density of ozone between the heights y_1 and y_2, and none elsewhere. In this case,

$$J_r = \frac{1}{s(f-1)}\{1 - e^{-(f-1)sm_1}\} + \int_{m_1}^{m_2} \exp\left[-(f-1)sm - \frac{a}{y_1 - y_2} w_{r, y_1, y, z}\right] dm$$
$$+ \frac{1}{s(f-1)} \exp\left[-\frac{a}{y_1 - y_2}\{w_{r, y_1, y', z} - w_{r, y_2, y', z}\}\right][e^{-s(f-1)m_2} - e^{-s(f-1)}],$$

where, in the last line, y' denotes some value of y between 0 and y_2.

7—Numerical Calculations

Extensive numerical calculations have been made to show how J and J/J′ or I/I′ depend on the ozone distribution. In almost all these calculations one set of values of a, s, a', s', have been used, viz.,

$$a = 0\cdot9025,\ a' = 0\cdot0784;\ s = 1,\ s' = 0\cdot8;$$

these are similar to those occurring in the paper by Götz, Meetham and Dobson, though not exactly equal to them.* Similar results would be expected from any neighbouring set of values. The calculations for any other set would need to be done independently. The values a, s refer to a wave-length inside the ozone absorption band, and the values a', s' to a longer wave-length near the limit of the band.

In the present section most of the results refer to a plane earth ($r = \infty$).

7.1. *Constant concentration of ozone; plane earth*—In this case, by 4.1,

$$R_\infty = \frac{a' + s'}{a + s}\,\frac{1 - e^{-Z(a+s)}}{1 - e^{-Z(a'+s')}};$$

this steadily decreases from 1, when $z = 0$, $Z = 0$, to $(a' + s')/(a + s)$ when $z = 90°$, $Z = \infty$; R_∞ has no minimum for this distribution of ozone.

For the above values of a, s, a', s', the limit $(a' + s')/(a + s)$ is $0\cdot46171$. The calculated values of $I_{\infty, z}/I_0$, $I'_{\infty, z}/I'_0$ and of R_∞ are given in Table II.

* Their values (for Arosa, where the mean barometric pressure is 609 mm), for the wave-lengths $\lambda = 3110$, $\lambda' = 3290$, are $s = 0\cdot656$, $s' = 0\cdot518$, and $a = 2\cdot994x$, $a' = 0\cdot2418x$, where x denotes the total amount of ozone expressed in cm of gas at N.T.P. Thus for $x = 0\cdot3$ cm, $a = 0\cdot898$, $a' = 0\cdot0725$.

TABLE II

z	0°	30°	60°	75°	80°	85°	90°
I/I_0	1	0·7580	0·2795	0·0975	0·0569	0·0253	0
I'/I'_0	1	0·8181	0·4159	0·1949	0·1213	0·0548	0
R_∞	1	0·9265	0·6720	0·5001	0·4688	0·4618	0·4617

Similar calculations have been made for the values $a + s = 1{\cdot}6$, $a' + s' = 0{\cdot}8$, for which $(a' + s')/(a + s) = \frac{1}{2}$.

The results for R_∞ are given in Table III (the separate values of I/I'_0 and I'/I'_0 are omitted).

TABLE III

z	0°	30°	60°	75°	80°	85°	90°
R_∞	1	0·9418	0·7247	0·5506	0·5111	0·50011	0·5

7.11. *Constant ozone concentration; curved earth; uniformly exponential atmosphere*—In this case (*cf.* 6.21) calculations have been made only for the values of $a + s$, $a' + s'$ last mentioned: two values of r have been considered, viz., 800 and 400. The values found for R_r are given in Table IV.

TABLE IV

z	0°	30°	60°	75°	80°	85°	90°	95°
R_{800}	1	0·9420	0·7260	0·5532	0·5131	0·50033	0·50000	0·50000
R_{400}	1	0·9421	0·7273	0·5558	0·5150	0·50064	0·50000	0·50000

While R_r differs appreciably from R_∞ at 75°, at $z = 90°$ the limiting value $\frac{1}{2}$ is attained very closely for the curved earth. The curvature has a far greater influence on I/I_0 and I'/I'_0 than on R_r; this is shown by the values given in Table V.

TABLE V

z		80°	85°	90°	z	80°	85°	90°
I/I_0	($r = \infty$)	0·0676	0·0301	0	I'/I'_0	0·1323	0·0601	0
I/I_0	($r = 800$)	0·0707	0·0344	0·0091	I'/I'_0	0·1378	0·0688	0·0181
I/I_0	($r = 400$)	0·0734	0·0378	0·0130	I'/I'_0	0·1426	0·0755	0·0259

7.2. *Uniform exponential ozone distribution*, $k = 2$; *plane and curved earth*—In this and all the later sections the calculations relate to the values of a, s, a', s' given in 7.

The present case corresponds to an ozone atmosphere much more strongly concentrated towards the ground than that of 7.1. The mean height of the ozone in that case was H, while in the present case it is $\frac{1}{2}$H.

The calculations refer to $r = \infty$ (4.2) and $r = 800$, $r = 400$ (6.22).

TABLE VI

$I_{r,z}/I_0$

z	0	30°	60°	75°	80°	85°	90°	95°
$r = \infty$	1	0·7759	0·3229	0·1348	0·0860	0·0423	0	—
800	1	0·7761	0·3240	0·1370	0·0889	0·0474	0·0136	$0{\cdot}0^3302$
400	1	0·7763	0·3251	0·1390	0·0916	0·0513	0·0190	$0{\cdot}0^2219$

$I'_{r,z}/I'_0$

z	0	30°	60°	75°	80°	85°	90°	95°
$r = \infty$	1	0·8198	0·4213	0·2017	0·1275	0·0588	0	—
800	1	0·8200	0·4224	0·2051	0·1326	0·0670	0·0180	$0{\cdot}0^3425$
400	1	0·8201	0·4235	0·2082	0·1372	0·0733	0·0256	$0{\cdot}0^2285$

R_r

z	0	30°	60°	75°	80°	85°	90°	95°
$r = \infty$	1	0·9464	0·7665	0·6685	0·6740	0·7192	0·8	—
800	1	0·9465	0·7670	0·6679	0·6704	0·7075	0·7577	0·7773
400	1	0·9466	0·7675	0·6675	0·6675	0·6994	0·7433	0·7672

In this case the limiting value of R_r is 0·8, attained at 90° for a flat earth, and approached, though much less closely than in 7.11, for the curved earth. In the present case R_r has a minimum value, such as the actual observations reveal, though it occurs at a different zenith-distance, as is natural since the actual ozone distribution is certainly not of the type to which the above figures relate.

PEKERIS (*loc. cit.*) gave what purported to be a proof that such a minimum value was impossible whatever the ozone distribution might be. The present case gives a direct disproof of this theorem, interesting because the present ozone distribution gives a particularly readily calculable formula for I/I_0 and I'/I_0'. An analytical proof of the existence of the minimum in this case could easily be given, but is unnecessary.

7.21. The proof by PEKERIS that R_r cannot have a minimum depends essentially on the supposition that the quantity

$$\frac{\int f(x)\,\phi(x)\,dx}{\int f'(x)\,\phi'(x)\,dx} - \frac{\int f(x)\,dx}{\int f'(x)\,dx}$$

((8), p. 8 of his paper) is positive, if all four functions f, f', ϕ, ϕ' are positive, and $\phi(x) > \phi'(x)$ for all values of x. The limits of integrations are supposed to be the same throughout. The functions f, f', ϕ, ϕ' are, in his paper, of definite form, but involve the ozone and air densities as functions of height: it appears that the above expression was supposed to be positive merely by virtue of the general conditions stated above, independently of their particular form. The following example shows that this is not so: let

$$f(x) = e^{-mx}, f'(x) = e^{-m'x}, \phi(x) = e^{-nx}, \phi'(x) = e^{-n'x},$$

where $n' > n$, so that $\phi(x) > \phi'(x)$. In this case the above expression, taking the limits of integration to be 0 and ∞, is equal to

$$\frac{mn' - m'n}{m(m+n)},$$

which may be positive or negative (with $n' > n$) according to the values of m and m'.

This of course does not in itself establish the existence of a minimum, but merely leaves the possibility open;* 7.1, 7.2 show that it may or may not be realised, according to the nature of the ozone distribution.

7.3. *Exponential ozone distributions above* $y = 3$; *plane earth*—Calculations based on the formulæ of 4.33 have been made for an exponential ozone distribution above the level $m = e^{-3}$, or, if the air is supposed uniformly exponential, above $y = 3H$ (*i.e.*, a height of 24 km if H = 8 km or 30 km if H = 10 km) for the cases $k = 1$, $k = 2$. The ozone is supposed absent below $y = 3$.

7.31. *Constant ozone concentration above* $y = 3$, $m = e^{-3}$, $k = 1$.

TABLE VII

z	0°	30°	60°	75°	80°	85°	90°
I/I_0	1	0·7078	0·1680	0·0205	$0{\cdot}0^2 6750$	$0{\cdot}0^2 2517$	0
I'/I'_0	1	0·8134	0·3989	0·1696	0·09462	0·03181	0
R_∞	1	0·8702	0·4211	0·1211	0·07134	0·07913	0·12415

It is of interest to note that R_∞ has a minimum in this case, although the ozone concentration is constant above $y = 3$ (and is zero elsewhere); *cf.* 7.51.

The mean height of the ozone in this case is 4H.

7.32. *Upward decreasing ozone-concentration above* $y = 3$, $m = e^{-3}$, $k = 2$.

TABLE VIII

z	0°	30°	60°	75°	80°	85°	90°
I/I_0	1	0·7141	0·1742	0·0268	0·01273	$0{\cdot}0^2 743$	0
I'/I'_0	1	0·8190	0·4032	0·1737	0·0989	0·0366	0
R_∞	1	0·8719	0·4321	0·1545	0·1287	0·2034	0·8

The mean height of the ozone in this case is 3·5H.

The striking features shown in comparing these two sets of results are their similarity up to $z = 75°$, and their great difference in the range 85° to 90°. The mode of variation of the ozone concentration at great heights greatly affects the limiting value of R_∞ as $Z \to \infty$, the limit being $(a'e^3 + s')/(ae^3 + s)$ when $k = 1$, and s'/s or 0·8 when $k = 2$. In the case of a curved earth the difference between the results for $k = 1$ and $k = 2$ would occur at zenith distances somewhat exceeding 85°.

* GAUZIT, 'C. R. Acad. Sci. Paris,' vol. 198, p. 1800 (1934), has already criticized the argument of PEKERIS on the ground of insufficiency; see also GÖTZ, 'Z., Astrophys.,' vol. 8, p. 267 (1934).

The similarity in the values of R_∞ up to 60° may be regarded as an instance of the considerable dependence of R_r on the mean height $\bar{y}$ of the ozone (which in the present instance is nearly the same in the two cases), which was remarked on by Götz, Meetham and Dobson; *cf.* also 7.1, Table II, corresponding to the same a, s, a', s', but to $\bar{y} = 1$, with Table VII of 7.31, for $\bar{y} = 4$, showing the change in R_∞ when $\bar{y}$ is much changed.

7.4. *Constant ozone density between $y = 5$ and $y = 2$; plane earth*—Calculations based on the formula 4.43 (1) have been made, taking $y_1 = 5$, $y_2 = 2$, and corresponding to a mean height $\bar{y} = 3{\cdot}5$ H, the same as that of the very different distribution considered in 7.32.

Table IX

z	0°	30°	60°	75°	80°	85°	90°
I/I_0	1	0·7098	0·1728	0·02407	$0{\cdot}0^{2}924$	$0{\cdot}0^{2}467$	0
I'/I'_0	1	0·8135	0·3997	0·1713	0·0967	0·0343	0
R_∞	1	0·8724	0·4323	0·1405	0·0956	0·1361	0·8

The values of R_∞ in this case show very close agreement with those of 7.32 up to $z = 75°$, and have the same limit at $z = 90°$. Only round about 85° is the disagreement notable. It is clear that decidedly accurate values of R_r for high zenith angles may be necessary to distinguish between even widely different ozone distributions corresponding to the same mean height $\bar{y}$.

7.41. *Constant ozone density below heights $y = 5$, $y = 3$ or $y = 1$; plane earth*—The results (Tables X–XII) are derived from calculations based on the formula 4.43 (4), and refer to mean heights $\bar{y}$ respectively equal to 2·5 H, 1·5 H and 0·5 H.

7.411. ρ_o constant below $y = 5$; $\rho_o = 0$ for $y > 5$; $\bar{y} = 2{\cdot}5$ H.

Table X

z	0	30°	60°	75°	80°	85°	88°	89°	90°
I/I_0	1	0·7259	0·2048	0·03985	0·01631	$0{\cdot}0^{2}616$	$0{\cdot}0^{2}3739$	$0{\cdot}0^{2}3045$	0
I'/I'_0	1	0·8151	0·4054	0·1798	0·1057	0·0416	0·013100	$0{\cdot}0^{2}6236$	0
R_∞	1	0·8906	0·5053	0·2217	0·1544	0·1481	0·2854	0·4883	0·8

Here the values of R_∞ exceed those of 7.4, corresponding to $y = 3{\cdot}5$ H, up to 85°, but R_∞ has the same limit at 90°; hence near 90° the two distributions, which both correspond to no ozone above $y = 5$H, seem likely to give nearly equal values of R_∞.

7.412. ρ_o constant below $y = 3$, $\rho_o = 0$ for $y > 3$; $\bar{y} = 1{\cdot}5$ H.

Table XI

z	0	30°	60°	75°	80°	85°	88°	89°	90°
I/I_0	1	0·7387	0·2375	0·0701	0·0425	0·0251	0·01424	$0{\cdot}0^{2}8425$	0
I'/I'_0	1	0·8163	0·4098	0·1872	0·1199	0·0509	0·02033	0·010737	0
R_∞	1	0·9049	0·5795	0·3746	0·3544	0·4928	0·7006	0·7847	0·8

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Here the values of R_∞ are greater than in the preceding case, except at 90° where the limit is the same: this corresponds to the lower value of y. Except near 90° the present values of R_∞ are fairly close to those (Table II) of 7.1, corresponding to $\bar{y} = H$, but at 90° the limiting values of R are different, because in 7.1 the ozone has constant concentration up to all heights.

7.413. ρ_o constant below $y = 1$; $\rho_o = 0$ for $y > 1$; $\bar{y} = 0 \cdot 5$ H.

TABLE XII

z	0°	30°	60°	75°	80°	85°	88°	89°	90°
I/I_0	1	0·7727	0·3222	0·1433	0·0955	0·0474	0·01810	$0 \cdot 0^2 888$	0
I'/I'_0	1	0·8195	0·4207	0·2018	0·1282	0·0596	0·02262	0·01110	0
R_∞	1	0·9429	0·7658	0·7103	0·7449	0·7947	0·80040	0·80000	0·8

This case refers to the same mean height as the exponential distribution ($k = 2$) of 7.2; up to $z = 60°$ the values of R are very similar for the two distributions, but beyond that they are higher in the present case, though the limit at 90° is the same for each.

7.5. *Ozone density constant below $y = 3$, exponential above; plane earth*—The following calculations are based on the formulæ of 4.42, and refer to the two cases $k = 1$ (constant ozone concentration above $y = 3$, and $\bar{y} = 2 \cdot 125$ H) and $k = 2$ (decreasing ozone concentration above $y = 3$, and $\bar{y} = 1 \cdot 786$ H).

7.51. $k = 1$; $y = 2 \cdot 125$ H.

TABLE XIII

z	0°	30°	60°	75°	80°	85°	90°
I/I_0	1	0·7308	0·2167	0·0493	0·0232	$0 \cdot 0^2 885$	0
I'/I'_0	1	0·8156	0·4071	0·1826	0·1089	0·0450	0
R_∞	1	0·8960	0·5323	0·2701	0·2130	0·1965	0·2158

It is of interest to note that R_∞ has a minimum in this case, although above $y = 3$ the ozone concentration is constant; *cf.* 7.1.

7.52. $k = 2$; $\bar{y} = 1 \cdot 786$ H.

TABLE XIV

I/I_0	1	0·7398	0·2296	0·0614	0·0340	0·01746	0
I'/I'_0	1	0·8215	0·4123	0·1882	0·1148	0·05109	0
R_∞	1	0·9006	0·5569	0·3264	0·2965	0·3418	0·8

As in previous cases, R_∞ is greater for the lower mean height; and also, near 90°, R_∞ depends largely on the ozone concentration at high levels.

7.6. *Possibility of a maximum value of R_r*—It has been seen that R_r may have a minimum value, and it might be supposed that in such cases it would approach its limiting value (as $Z \to \infty$) from below. An example to the contrary has, however, already been given, for $r = \infty$, in 7.413, which shows that R may have a maximum following the minimum, and therefore may approach its limit from above; the R, z curve for 7.413 seems, moreover, to be horizontal at $z = 90°$.

Another example is provided by calculations made from the formula of 6.41, taking $y_1 = 3$, $y_2 = 0$, so that the ozone distribution is the one dealt with in 7.412; but the limitation to a plane is here removed, and the results refer to $r = 800$. Only two zenith-distances have been considered, viz., 88° and 92°.

TABLE XV

z	88°	92°
$I_{800,z}/I_0$	0·1797	$0{\cdot}0^2518$
$I'_{800,z}/I'_0$	0·3063	$0{\cdot}0^2642$
R_{800}	0·5865	0·8074

Thus at $z = 92°$ R_{800} has risen above its limiting value, to which it must thereafter descend.

7.7. *The zenith distance for minimum R_∞*—Table XVI gives the zenith distance $z_{\min}$* at which R_∞ attains its minimum value ($R_{\min}$), for the various ozone distributions on a plane earth already considered and for the set of values of a, s, a', s' referred to in 7. Besides $z_{\min}$, the table gives $R_{\min}$, and also R_∞ (75°) and R_∞ (90°) (the limiting value of R_∞). The cases are arranged in ascending order of $\bar{y}$, the mean height of the zone.

Where two distributions have the same $\bar{y}$, they are placed in descending order of R_∞ (75°).

TABLE XVI

Reference	Ozone distribution	$\bar{y}$	R_∞(75°)	$R_{\min}$	R_∞(90°)	$z_{\min}$
7.413	ρ_o const., $y < 1$; $\rho_o = 0$, $y > 1$	0·5	0·710	0·71	0·8	73°
7.2	$\rho_o \propto \rho^2$	0·5	0·669	0·66	0·8	78
7.1	ρ_o/ρ constant	1	0·500	0·46	0·462	90
7.412	ρ_o const., $y < 3$; $\rho_o = 0$, $y > 3$	1·5	0·375	0·35	0·8	79
7.52	ρ_o const., $y < 3$; $\rho_o \propto \rho^2$, $y > 3$	1·79	0·326	0·296	0·8	81
7.51	ρ_o const., $y < 3$; $\rho_o \propto \rho$, $y > 3$	2·12	0·270	0·194	0·216	84
7.411	ρ_o const., $y < 5$; $\rho_o = 0$, $y > 5$	2·5	0·222	0·135	0·8	83
7.32	$\rho_o \propto \rho^2$, $y > 3$; $\rho_o = 0$, $y < 3$	3·5	0·154	0·130	0·8	80
7.4	ρ_o const., $2 < y < 5$; $\rho_o = 0$, $y < 2, y > 5$	3·5	0·140	0·093	0·8	81·5
7.31	$\rho_o \propto \rho$, $y > 3$; $\rho_o = 0$, $y < 3$	4	0·121	0·067	0·124	81·5

The most important and interesting feature of Table XVI is that R_∞ (75°) and $R_{\min}$ steadily decrease as $\bar{y}$ increases. The regularity in this respect is in striking contrast with the irregularity in the values of $z_{\min}$ and of R_∞ (90°). But $R_{\min}$ does not depend only on $\bar{y}$, because different distributions having the same value of $\bar{y}$ give different values of $R_{\min}$; *i.e.*, for $\bar{y} = 0{\cdot}5$, the Table gives two values of $R_{\min}$, 0·71 and 0·66; likewise, for $\bar{y} = 3{\cdot}5$, the values 0·13 and 0·09 for $R_{\min}$. The differences in $R_{\min}$ exist in spite of the fact that for these particular pairs of distributions R_∞ (90°) has the same value. In both pairs, however, the smaller value of $R_{\min}$ corresponds to the larger value of $z_{\min}$.

* Read from the graphs of the figs. on p. 229.

7.8. All the preceding calculations are illustrated graphically in fig. 1, in which R is plotted against z^4 (as in fig. 1 in the paper by Götz, Meetham and Dobson). In this figure R_∞ is indicated by full lines, and R_r by dotted lines.

The curves are numbered downwards, and in the same diagram the ozone distributions to which the curves refer are also indicated.

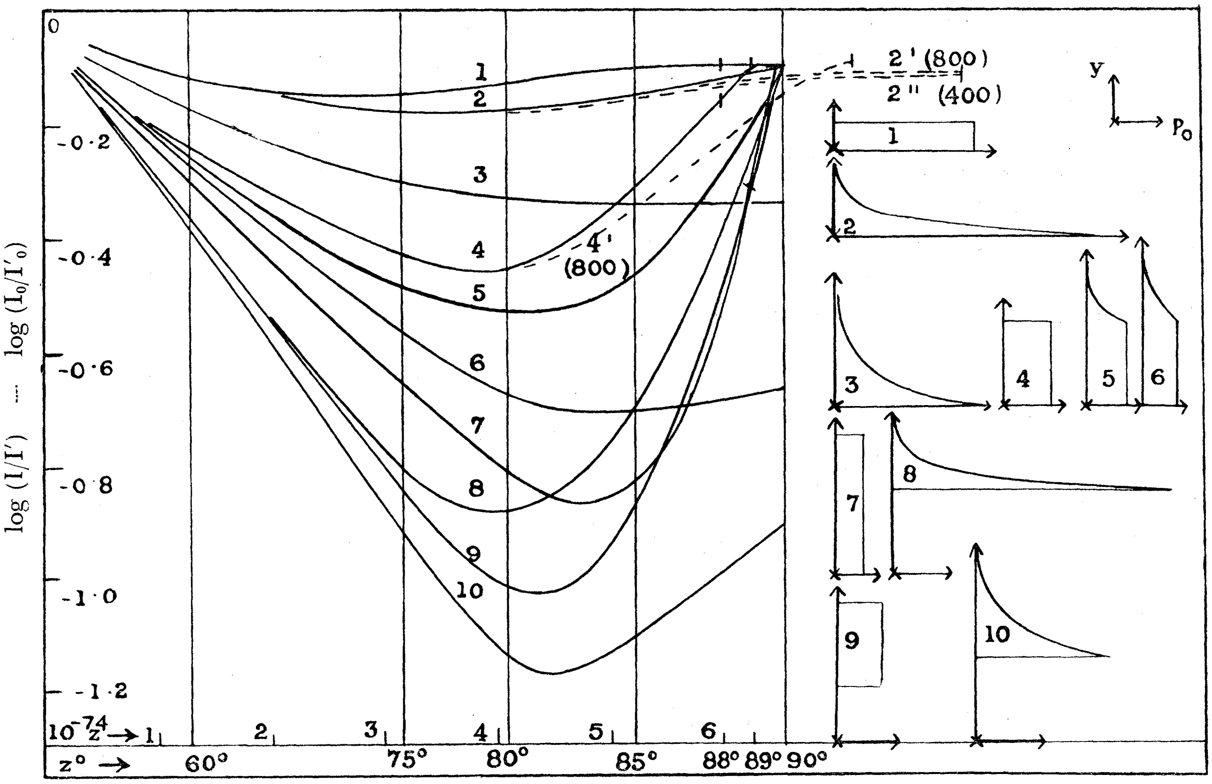

Fig. 1

8—Comparison with Observation

The present paper is intended to give a general survey of the R_r, z relation for a number of different atmospheric and ozone distributions that are susceptible to exact treatment up to a fairly late stage of the analysis. The methods used can be applied to several more elaborate distributions, without difficulty other than that arising from the extra complexity of the formulæ.

It is not the purpose here to discuss particular observations of R_r in order to determine the ozone distribution. Many difficult problems arise in such determinations, which it is hoped to discuss in a further paper. A few general remarks on this subject will, however, be added.

Consider a given R, z curve for a known value of M_o. It is desirable first of all to ignore the curvature of the earth, and to discuss the observed curve as if it were for a plane earth; except that, assuming the value of R_∞ (90°) to be s'/s or σ'/σ,

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the observed curve should be slightly contracted along the direction of the z axis, near 90°, so that it tends to this limit at 90°. A little experience will probably enable this to be done with considerable accuracy. The value of R_{min} will give an approximate estimate for $\bar{y}$, and the R, z curves for a few likely ozone distributions, for this and neighbouring values of $\bar{y}$, can then be calculated. This will enable one to find the range of possible distributions, and perhaps also the best distribution (D), to fit the observed (adjusted) R, z curve. A final calculation of the R, z curve for the distribution D for the curved earth will then serve to check the suitability of this distribution, and, also, the accuracy of the initial adjustment of the observed curve. Even in this final calculation it will probably suffice to treat the atmosphere as uniformly exponential, taking a value of r corresponding to a mean value of H applicable over the range of height up to about 40 km. Afterwards, knowing the actual distribution of air density, the actual height-distribution of the ozone can be calculated without much trouble from the distribution D in the exponential atmosphere. It would certainly seem that the considerably different height-distributions of the air at times of large and small values of M_o should be taken into account when inferring the height-distribution of the ozone from the R, z curves.

It is hoped to apply these methods, by the courtesy of Dr. G. M. B. Dobson, to the re-discussion of the Arosa data already published, and also to data from Tromsø more recently obtained.*

It is hoped also to consider in detail the influence of secondary scattering upon the R, z curve, and to examine whether observations of scattered light from other parts of the sky than the zenith offer promise of usefully supplementing the zenith-sky observations.

I wish gratefully to acknowledge the help I have received in the preparation of the paper from Dr. J. C. P. Miller, Research Assistant in the Department of Mathematics at the Imperial College of Science and Technology. He executed all the detailed numerical calculations involved, and has also verified the numerous formulæ.

Summary

Formulæ are given for the intensity of zenith-scattered sunlight at different altitudes of the sun, for a wave-length in the ozone absorption band. These formulæ refer to special ozone-distributions, in an atmosphere on either a plane or a spherical earth. It is shown that in most cases, though not all, the ratio of the zenith-light, for two wave-lengths, has a minimum when the sun is low, thus agreeing with observation, but disagreeing with a supposed proof to the contrary, given by Pekeris. Numerical illustrations of the formulæ are given and briefly discussed.

* 'Proc. Roy. Soc.,' A, vol. 148, p. 598 (1935).